Engineering Notes

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A Simple Algorithm for the Selection of Terminal Penalty Weighting Matrices

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Introduction

ONE of the most basic tasks required in solving finite fixed-time linear-quadratic-regulator problems is the specification of a terminal penalty matrix. In theory, a matrix (frequently, a diagonal matrix) whose elements are large in an engineering sense is simply selected. However, the resulting control-gain histories often exhibit fluctuations of several orders of magnitude near the end of the control interval. Theoretically, this is not a problem because the states are close to zero when the fluctuations occur. Computationally, however, it is difficult to simulate the closed-loop system response when the control gains are varying rapidly. In an effort to overcome this problem, a simple algorithm for selecting the terminal penalty weighting matrix is presented in this Note; the algorithm minimizes the large fluctuations in the terminal values of the control gains while satisfying the terminal-state boundary conditions.

The basic approach consists of choosing the value of the Riccati matrix just prior to the time that large fluctuations occur as the candidate terminal weighting matrix, since at this time the desired boundary conditions for the state have generally been achieved. Based on this approach, an iterative algorithm is developed, which makes efficient use of closed-form solutions for the time-varying Riccati matrix and the closed-loop system response. The algorithm is iterative, since the magnitude of the starting guess for the terminal weighting matrix influences the time when the Riccati matrix undergoes rapid fluctuations, and the errors in the satisfaction of the terminal state boundary conditions.

The Optimal Control Problem

The optimal control problem is formulated by seeking the control inputs $u(t)$ to minimize

$$J=\frac{1}{2}x_f^T S_f x_f+\frac{1}{2}\int_0^{t_f}(x^TQx+u^TRu)\mathrm{d}t \tag{1}$$

subject to

$$\dot{x}=Ax+Bu, \qquad x(t_0)=x_0, \qquad x(t_f)\approx 0 \tag{2}$$

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The optimal control[1] is given by

$$u(t)=-R^{-1}B^TP(t)x(t) \tag{3}$$

where

$$\dot{P}=-PA-A^TP+PBR^{-1}B^TP-Q; \quad P(t_f)=S_f \tag{4}$$

A is the system dynamics matrix, B the control influence matrix, $Q=Q^T\geq 0$ the state weight matrix, $R=R^T>0$ the control weight matrix, $S_f=S_f^T\geq 0$ the terminal penalty weight matrix, and P the time-varying Riccati matrix. Furthermore, it is assumed that (A,B) is stabilizable and $(Q^{1/2},A)$ detectable.

The closed-form solution for the differential matrix Riccati equation is given by[2-6]

$$P(t)=P_{ss}+Z^{-1}(t) \tag{5}$$

where P_{ss} satisfies the algebraic Riccati equation

$$0=-P_{ss}A-A^TP_{ss}+P_{ss}BR^{-1}B^TP_{ss}-Q$$

The closed-form solution for $Z(t)$ in Eq. (5) can be shown to be

$$Z(t)=Z_{ss}+e^{\tilde{A}(t-t_f)}[(S_f-P_{ss})^{-1}-Z_{ss}]e^{\tilde{A}^T(t-t_f)} \tag{6}$$

where $\tilde{A}=A-BR^{-1}B^TP_{ss}$ is the system stability matrix, $e^{(\cdot)}$ is the exponential matrix, $S_f\neq P_{ss}$, and Z_{ss} satisfies the algebraic Lyapunov equation $\tilde{A}Z_{ss}+Z_{ss}\tilde{A}^T=BR^{-1}B^T$.

Subject to the previously listed assumptions for the system considered in this note, it follows that P_{ss}, Z_{ss}, $Z(t)$, and $Z^{-1}(t)$ exist for $0\leq t<T$.

Computational Algorithm

To begin, a starting guess for the terminal penalty matrix is selected

$$S_f=\bar{S}_f \tag{7}$$

In order to find the time when the Riccati matrix changes rapidly, a step size Δt is defined for propagating the Riccati solution backwards in time. The authors have chosen Δt to be the same order of magnitude as the step size used for simulating the closed-loop control.

The recursion relation for propagating the Riccati matrix of Eq. (5) at time steps equal to $(-\Delta t)$ follows as

$$P_k=P_{ss}+Z_k^{-1} \qquad (k=1,2,3...) \tag{8}$$

where

$$P_1=P_f=S_f,\ Z_1=(S_f-P_{ss})^{-1}$$

and

$$Z_k=e^{-\tilde{A}\Delta t}[Z_{k-1}-Z_{ss}]e^{-\tilde{A}^T\Delta t}+Z_{ss}$$

To find the time when the Riccati matrix changes rapidly, we compare the Riccati solution at two neighboring times. In particular, after each propagation step k, each element of P_k is compared with the corresponding element of P_{k-1} for the satisfaction of the following inequality:

$$1/h < (P_{k-1})_{ij}/(P_k)_{ij} < h, \quad (P_k)_{ij} \neq 0$$

$$|(P_{k-1})_{ij}| < h, \quad (P_k)_{ij} = 0 \tag{9}$$

where $h \approx 10$, $i = 1,...n$, $j = 1,...i$, and n is the number of states. The criterion of Eq. (9) requires that during the interval Δt the change in each element of the Riccati matrix must be less than a factor of h.

The earliest propagation step k for which the above condition holds provides the candidate terminal weight matrix:

$$S_f = P_k \tag{10}$$

where P_k is obtained using Eq. (8).

The time histories for the closed-loop system response follow from integrating the system dynamics equation

$$\dot{x}(t) = [A - BR^{-1}B^T P(t)]x(t), \quad x_0 = x(t_0) \tag{11}$$

where $P(t)$ is given by Eq. (5). As shown in Ref. 7, the closed-form solution for Eq. (11) is given by

$$x(t) = \phi(t,t_0)x_0 \tag{12}$$

where

$$\phi(t,t_0) = Z(t)e^{-\tilde{A}^T(t-t_0)}Z^{-1}(t_0)$$

is the state transition matrix and $Z(t)$ is defined by Eq. (6), where $\phi(t_2,t_0) = \phi(t_2,t_1)\phi(t_1,t_0)$, and $\phi^{-1}(t_1,t_0) = \phi(t_0,t)$.

The state trajectory is computed using the terminal penalty matrix provided by Eq. (10) and Eq. (12). If all the final states are within the desired bounds, i.e.,

$$|x_i(t_f)| < \epsilon_i \qquad i = 1,2,...n \tag{13}$$

where ϵ_i is a preassigned convergence tolerance for the state, then the converged terminal weight matrix of Eq. (10) is the desired weight matrix. However, if Eq. (13) is not satisfied for all the states, then the $\bar{S}_f$ matrix of Eq. (7) is modified by increasing the penalty on those states for which Eq. (13) is not satisfied, as follows:

$$[\bar{S}_f]_{ii} = [\bar{S}_f]_{ii} H[x_i(t_f),\epsilon_i] \tag{14}$$

where

$$H[x_i(t_f),\epsilon_i] = \begin{cases} 1 \text{ if } |x_i(t_f)| \leq \epsilon_i \\ 10 \text{ if } |x_i(t_f)| > \epsilon_i \end{cases}$$

The modified $\bar{S}_f$ matrix is then used in Eq. (7) as the new starting guess for the terminal weight.

The imposition of the criterion of Eq. (9) has been found to ensure that the elements of the Riccati matrix do not change rapidly near the final time for the control problem. Many different starting guesses, $\bar{S}_f$, may be used. For example, a purely diagonal matrix may be used for $\bar{S}_f$. Another starting guess is the sum of the steady state Riccati matrix P_{ss} and a diagonal matrix [which may be considered a starting guess for $Z^{-1}(t_f)$]. Still another approach is to use a steady-state Riccati matrix multiplied by a large number, where the steady-state Riccati matrix may correspond to a steady-state controller with short settling times. In addition, another variation that can be used is the previous converged P_k of Eq. (10) on the right-hand side of Eq. (14) to replace $\bar{S}_f$ when computing the starting guess for the next iteration. In some cases, this leads to quicker convergence of P_k.

Nevertheless, the algorithm of this paper only considers modifying the terminal weight matrix, S_f of Eq. (1). It is clear, however, that a complete treatment of the problem requires that the elements of the Q and R matrices should also be refined. The techniques necessary for accomplishing this objective have recently been considered by Junkins, Bodden, and Turner.[8]

Example

As an example case, the algorithm is used to select the terminal weights for the control problem

$t_f = 5$ s, $x(0) = [10,0,0,0]^T$

$\epsilon = [0.1, 0.06, 0.06, 0.06]^T$, $B = [0,0,0,1]^T$, $R = [1]$

$Q = \text{diag}[1(-3), 1(-2), 1(-9), 1(-9)], 1 \;\; a(b) = a \times 10^b$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -0.01 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{ss} = \begin{bmatrix} 1.309 & -.0275 & -.8956 & -.2548 \\ -.0275 & 1.343 & .8022 & .1350 \\ -.8956 & .8022 & 13.04 & 3.256 \\ -.2548 & .1350 & 3.256 & 4.045 \end{bmatrix}$$

The parameters chosen for the algorithm are $\Delta t = 0.001$ s, $h = 10$.

Case 1

For a diagonal starting guess given by $\bar{S}_f = \text{diag}[0.1, 0.1, .001, .001]$. The converged terminal weight matrix (after seven iterations) is

$$S_f = \begin{bmatrix} 1000 & 3.998(-5) & -2.000(-3) & -2.000(-6) \\ 3.998(-5) & 1000 & 2.000 & .001556 \\ -2.000(-3) & 2.000 & 1000 & 1.333 \\ -2.000(-6) & .001556 & 1.333 & 333.3 \end{bmatrix}$$

The maximum eigenvalue of the converged S_f is 1002. The final states are $x(t_f) = [.0146, .0154, -.0249, .0295]^T$.

Case 2

For a starting guess given by the sum of P_{ss} and a diagonal Z_f^{-1}, where $Z_f^{-1} = \text{diag}[1(-3), 1(-3), 1(-3), 1(-3)]$. The converged terminal weight matrix (after nine iterations) is

$$S_f = \begin{bmatrix} 1000 & -8.826(-5) & -0.1035 & -0.01193 \\ -8.826(-5) & 1000 & 2.002 & -0.02794 \\ -0.1035 & 2.002 & 1000 & 1.356 \\ -0.01193 & -0.02794 & 1.356 & 333.4 \end{bmatrix}$$

The maximum eigenvalue of the converged S_f is 1002, and the final states are $x(t_f) = [.0146, .0154, -0.0249, .0295]^T$.

It is seen that the results from the two starting guesses are quite close to each other. For the starting guess which consists of a steady-state Riccati matrix multiplied by a large number, the resulting final states (not shown) are much smaller; however, the elements of the converged S_f and its eigenvalues are much larger, which makes this starting guess less useful.

Test cases were also run for problems in which the integral term in Eq. (1) included penalties involving products of x and u. Such penalty terms were found to be necessary for the states to converge in some control smoothing problems.

Conclusions

A simple iterative algorithm has been presented for selecting terminal penalty weighting matrices for linear-quadratic control problems. The algorithms selects a weighting matrix which yields acceptable system performance, while avoiding the large changes in magnitude in final values of the feedback gains. The algorithm is made computationally efficient by using closed-form solutions for the time-varying Riccati equation and the closed-loop system response. The method is heuristic, but offers a systematic approach to terminal weight selection.

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Comparison of Angular and Metric Guidance Laws for Tactical Missiles

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Introduction

In 1979, Nesline and Zarchan[1] compared classical and modern homing guidance for tactical missiles. The classical, or proportional navigation (PN), laws correspond to the application of a filter to the missile-to-target line-of-sight (LOS) rotation rate. Guidance based on such laws can be termed "angular." A useful analytical tool in this context is the adjoint technique,[2,3] first introduced by Bennet and Mathews[4] and used by Nesline and Zarchan[5] to determine the performance of a given guidance loop. However, there is nothing in the open literature concerning the use of this technique for the synthesis of an optimal filter, which is possible.

The laws describing modern guidance systems are based on the methods of optimal control, which have been suitably summarized by Bryson and Ho.[6] Invoking the separation theorem, these laws are presented as comprising an estimator and a deterministic controller in cascade. These laws can be termed "metric" if they process the Cartesian coordinates of the missile-to-target vector. The miss distance can be minimized using many possible terminal control criteria,[7] or its estimate can be forced to zero.[8]

This Note compares an angular guidance law optimized by the adjoint technique and a metric law comprising an optimal estimator connected to a terminal controller forcing the estimated miss distance to zero. The terms of comparison are miss distance and realizability. An analytical solution is given for one specific but significant case. To the author's knowledge, this is the first time such a solution has been published.

Optimal Metric Law

The guidance model is unidimensional and linearized about the mean collision line. It is assumed that the time to go, $t_{go} = t_f - t$, and the relative missile-to-target velocity, V_c ($V_c > 0$), are measured. The corresponding equations are:

$$\ddot{y} = a_T, \qquad \ddot{y}_M = a_M \tag{1}$$

$$y_M - y_T = -\sigma V_c t_{go} + n \tag{2}$$

where y_M and y_T are the missile and target coordinates relative to the mean collision line, σ the angle between the LOS and the mean collision line, a_M and a_T the missile and target lateral accelerations, and n denotes the zero-mean metric noise. Further,

$$a_M = P(s) a_c \tag{3}$$

where a_c is the missile commanded lateral acceleration and $P(s)$ the autopilot transfer function.

The optimal control law comprises an optimal state estimator dependent upon measurement noise n and process noise a_T, connected to a terminal controller which, in the absence of noise, forces the miss distance $y_{TM}(t_f) = y_M(t_f) - y_T(t_f)$ to zero.

$$a_c(t) = C(t)\hat{y}(t) \tag{4}$$

$\hat{y}(t)$ denotes the estimate of the state vector $y(t)$. The components of this vector are $y_{TM}(t)$, $\dot{y}_{TM}(t)$, and other variables depending on the manner in which a_T is modeled. If no limitations are placed on lateral acceleration, then the miss distance, σ_d, depends solely on measurement noise and target maneuvering (TM).

$$\sigma_d^2 = E[\tilde{y}^2_{TM}(t_f, n, a_T)] \tag{5}$$

Assuming that measurement noise and target maneuvering are stationary with spectral densities $\phi_{nn}(s) = N(s)N(-s)$ and $\phi_{TT}(s) = T(s)T(-s)$, then the asymptotic estimator minimizing $\sigma_d^2 = E[\tilde{y}^2_{TM}(\infty)]$ is a Wiener filter with transfer function $W(s)$, as indicated by the diagram illustrated in Fig. 1. The corresponding equation is

$$\sigma_d^2 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left[\phi_{TT}(s)\left(\frac{1-W(s)}{s^2}\right)\left(\frac{1-W(-s)}{s^2}\right) + \phi_{nn}(s)W(s)W(-s)\right] ds \tag{6}$$

The quantity a_M is assumed to be known either by direct measurement or by simulating $P(s)$. This yields $\tilde{y}_{TM} = \tilde{y}_T$.

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